Universality and deviations in disordered systems

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We compute the probability of positive large deviations of the free energy *per spin* in mean-field spin-glass models. The probability vanishes in the thermodynamic limit as $P(\Delta f)\propto \exp[-N^2 L_2(\Delta f)]$. For the Sherrington–Kirkpatrick model we find $L_2(\Delta f)=O(\Delta f)^{12/5}$ in good agreement with numerical data and with the assumption that typical small deviations of the free energy scale as $N^{1/6}$. For the spherical model we find $L_2(\Delta f)=O(\Delta f)^3$ in agreement with recent findings on the fluctuations of the largest eigenvalue of random Gaussian matrices. The computation is based on a loop expansion in replica space and the non-Gaussian behavior follows in both cases from the fact that the expansion is divergent at all orders. The factors of the leading order terms are obtained resumming appropriately the loop expansion and display universality, pointing to the existence of a single universal distribution describing the small deviations of any model in the full-replica-symmetry-breaking class.

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I. VERY LARGE DEVIATIONS OF THE FREE ENERGY

The free-energy density of a random system model fluctuates over the disorder and is described by a distribution that in the thermodynamic limit becomes peaked around a typical value f_{typ}. The probability of observing $O(1)$ deviations (i.e., large deviations) from the typical value are exponentially small. In the spherical mean-field spin-glass model[1] the probability of large deviations is highly asymmetric. The probability of observing negative deviations $\Delta f = f - f_{typ}$ scales exponentially with the system size N, i.e., $P(\Delta f)\propto \exp[-NL(\Delta f)]$ with a function $L(\Delta f)$ that can be computed through a zero-loop computation.[2] Instead it can be argued that the probability of observing *positive* deviations is much smaller, it scales exponentially with the *square* of the system size, i.e., $P(\Delta f)\propto \exp[-N^2 L_2(\Delta f)]$ with an appropriate function $L_2(\Delta f)$. This function can be computed at zero temperature noting that the energy of the model is given by the lowest eigenvalue of a random $N\times N$ Gaussian matrix. The probability of positive large deviations of the lowest eigenvalue of a Gaussian random matrix has been recently computed in Ref. 3 and scale indeed exponentially with the square of N. On the other hand the *typical* small deviation distribution of the lowest eigenvalue are described by a universal function in the large N limit that was computed by Tracy and Widom[4] and has since appeared in many apparently unrelated problems (see, e.g., Ref. 3).

In this paper, we show that in the mean-field Sherrington–Kirkpatrick (SK) spin-glass model a similar scaling is present and the probability distribution $L_2(\Delta f)$ can be computed using the hierarchical replica-symmetry-breaking (RSB) ansatz. This result is complementary to the computation of the function $L(\Delta f)$ that is relevant for negative deviations and was computed in[2]). Interestingly enough our results also show a great deal of universality suggesting that the small deviations of the free energy of the whole class of full-RSB (fRSB) models are also described by the same universal function at any temperature in the spin-glass phase.

The computation of $L_2(\Delta f)$ is much more complex than that of $L(\Delta f)$, indeed the function $L(\Delta f)$ is infinite for positive Δf.[5,6] As we will see in the following, in the replica framework the problem is the computation of an integral over the components of a matrix $\tilde{Q}_{ab}$ of size $n\times n$ where $n=\alpha N$ with negative α, notice that while the standard replica trick requires to send the number of replicas to zero in this case the number of replicas has to be sent to minus infinity. Usually fluctuations around the saddle point are negligible (they give a contribution to the total free energy that is proportional to n^2), instead they play a crucial role in this case because the number of elements of the matrix $\tilde{Q}$ is $O(N^2)$. We have also to cope with the fact the saddle point is marginal (i.e., there are zero modes that lead to divergent fluctuations) thus if we proceed naively we obtain a series divergent at all orders. The marginality of mean-field theory is a general well-known feature of full-RSB spin-glass and it stands as the main difficulty to solve many of the open problems in the field. In particular it leads to nontrivial finite-size correction in the SK model for which no systematic computation scheme has been devised up to now.[7] Besides the fact that no theory can be considered complete if it does not allow to include deviations and corrections, these nontrivial mean-field exponents have been argued to be relevant also for finite-dimensional quantities, notably the stiffness exponent.[8] Furthermore nobody knows how to treat loop corrections to the theory in the spin-glass phase below six dimensions,[9] again because of the marginality of mean-field theory. Thus the need to tackle the marginal nature of the theory makes the computation as interesting as the problem itself.

The SK model is considered the paradigm of frustrated system, as such it is also an important testing ground for minimization algorithms aiming at finding the ground state of a given instance of size N. This adds further interest to the computation of finite-size corrections and large deviations to the thermodynamic limit estimates. The results of our computation, see Fig. 1, are indeed important to understand the behavior of the numerical data available through present-day technology that display important corrections to the thermodynamic limit behavior.

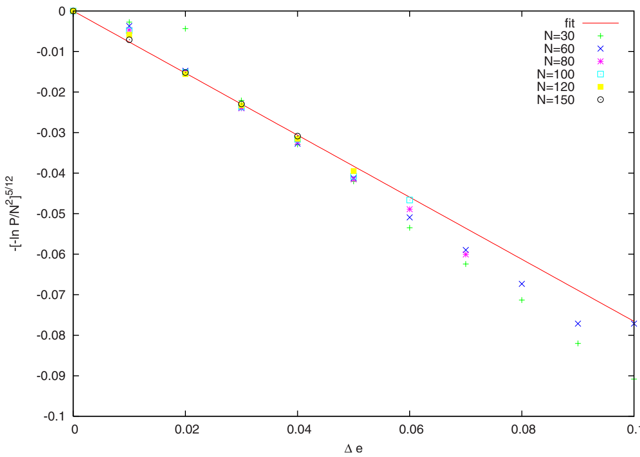

FIG. 1. (Color online) Numerical Sample Complexity vs Energy difference at zero temperature for the SK model (data of Ref. 10 courtesy of the authors). The data are rescaled with the exponent 5/12 and are well fitted by a linear function $-.765\Delta e$, which combined with the result $\lim_{T\to 0} T\dot{q}(0)=0.743\,368$ leads to $C_{fRSB}=-.64(2)$

An interesting issue is the behavior of the small deviations of the free energy. These are deviations that are observed with *finite* probability and their distribution gets peaked around the typical value f_{typ} as the system size increases. It is usually assumed that upon introducing a rescaled variable $\delta=c(f-f_N)N^a$ the small distribution $P(\delta)$ has a well defined limit in the thermodynamic limit, therefore typical deviations expected at system size N scale as N^{-a}. For random Gaussian matrices the small deviations of the smallest eigenvalue are described by the celebrated Tracy-Widom distribution.[4] On the other hand, the fact that in the SK model the negative large-deviation probability goes as $P(\Delta f)\propto\exp[-NL(\Delta f)]$ with $L(\Delta f)=O(\Delta f)^{6/5}$ motivated the early suggestion that small-deviations scale as $N^{-5/6}$.[11] The argument is very simple, one has just to rescale Δf with the appropriate power of N in such a way that $P(\Delta f)\propto\exp[-NL(\Delta f)]$ remains finite in the thermodynamic limit. The scaling of small deviations has been under a heated debate in recent years and there is increasing consensus on the fact that the correct exponent is indeed 5/6.[7] Interestingly enough the results presented here adds further support to it, indeed we find that the probability of positive large deviations in the SK model scales as $P(\Delta f)\propto\exp[-N^2L_2(\Delta f)]$ with $L_2(\Delta f)=O(\Delta f)^{12/5}$ and using the same argument we obtain again the exponent 5/6 for the small deviations. Furthermore, the leading exponent of the function $L_2(\Delta f)$ at small values of Δf turns out to depend just on the full-RSB or replica-symmetric (RS) nature of the model considered. In the first case we find $L_2(\Delta f)=O(\Delta f)^{12/5}$ while for RS models we find $L_2(\Delta f)=O(\Delta f)^3$. Thus, we conjecture that it exists a universal function describing the small-deviations of models with full-RSB, much as the Tracy–Widom laws describes those of RS models such as the spherical mean-field spin glass.

II. REPLICATED FUNCTIONAL FOR THE SAMPLE COMPLEXITY

In order to compute the so-called *sample complexity* $L_2(\Delta f)\equiv-\ln[P(\Delta f)]/N^2$ we consider the average partition function of $n=\alpha N$ replicas:

$$\Phi(\alpha)\equiv-\frac{1}{N^2}\ln\overline{Z^{\alpha N}}, \tag{1}$$

with α negative (the bar denotes the average over the disorder). In the usual approach, when we need to compute the typical free energy (f_{typ}) the number (n) of replicas goes to zero, but in this case it has to go to $-\infty$.

The functional $\Phi(\alpha)$ is the Legendre transform of $L_2(\Delta f)$. The latter can be obtained as $-L_2(\Delta f)=\alpha\beta f_{typ}+\alpha\beta\Delta f-\Phi(\alpha)$ where α is the solution of $\beta f_{typ}+\beta\Delta f=d\Phi(\alpha)/d\alpha$. The main result of this paper is the evaluation of $\Phi(\alpha)$ for small α.

Let us now consider what happens in the SK spin-glass model where $H=\sum_{i,k}J_{i,k}\sigma_i\sigma_k$ with σ's being the Ising spins and J's being the random Gaussian variable with zero average and variance $1/N$. Through standard manipulations we rewrite Eq. (1) as the integral of an action depending on a symmetric matrix $\tilde{Q}_{ab}$ with $a,b=1,\ldots,\alpha N$ and $\tilde{Q}_{aa}=0$. As the size of the matrix is $O(N)$ we cannot simply take the saddle point and we have consider the integral of $O(N^2)$ elements of $\tilde{Q}_{ab}$. In order to perform the integration we find convenient to divide the matrix $\tilde{Q}_{ab}$ in $(\frac{N\alpha}{n})^2$ blocks of size $n\times n$, with n some parameter between 1 and αN that will be eventually sent to zero. The generic matrix element will be written as Q^{ij}_{ab} where the upper indices $i,j=1,\ldots,N\alpha/n$ label different blocks and the lower indices $a,b=1,\ldots,n$ label elements inside block ij.

The resulting action admits a saddle point with vanishing off-diagonal blocks $Q^{ij}=0$ for all $i\neq j$ and we integrate out the elements of the off-diagonal blocks $i\neq j$ around their zero saddle-point value. After the integration we are left with an integral over the diagonal blocks of the exponential of a $O(N^2)$ action over which we will take the saddle point. We will consider here the simplest situation where the saddle points are such that all the blocks on the diagonal are equal to a given $n\times n$ hierarchical matrix Q, i.e., $Q^{ii}=Q\ \forall i$. At the end the function $\Phi(\alpha)$ will be obtained as the saddle-point value of the following functional over the $n\times n$ diagonal block Q,

$$\Phi(\alpha,Q)=\alpha\beta F[Q]+S[Q,\alpha], \tag{2}$$

where $F[Q]$ is the standard SK free energy functional of a $n\times n$ matrix Q_{ab} that is zero on the diagonal,

$$F[Q]=-\frac{\beta}{4}+\frac{\beta}{2n}\sum_{a<b}^{n}Q_{ab}^2-\frac{1}{\beta n}\ln\sum_{\{s\}}\exp\left[\beta^2\sum_{a<b}Q_{ab}s_as_b\right], \tag{3}$$

and $S[Q,\alpha]$ is the contribution of the fluctuations. Now $S[Q,\alpha]$ is small for small α and it should be treated as a perturbation: in order to obtain the first non trivial term we should compute $S[Q,\alpha]$ at the saddle point of $F[Q]$. This is not a easy task: indeed we find that $S[Q,\alpha]$ can be written as

$$S[Q,\alpha] = -\frac{\alpha^2}{N^2}\ln\int\left(\prod_{i<j}^{-N/n}\prod_{ab}^{n}\frac{dQ_{ab}^{ij}}{\sqrt{2\pi}}\right)\times\exp\left[-\frac{1}{2}\sum_{i<j}^{-N/n}\sum_{ab}^{n}(1-\beta^2\lambda_a\lambda_b+p)(Q_{ab}^{ij})^2+\left(\frac{-\alpha}{N}\right)^{1/2}\beta^3\sum_{i<j<k}^{-N/n}\sum_{abc}^{n}Q_{ab}^{ij}Q_{bc}^{jk}Q_{ca}^{ki}\lambda_a\lambda_b\lambda_c\right], \quad (4)$$

where λ_a with $a=1,\ldots,n$ are the eigenvalues of the matrix $P_{ab}\equiv\langle s_a s_b\rangle$ and the averages $\langle\cdot\rangle$ are computed with respect to a single diagonal block. In particular if Q_{ab} extremizes $F[Q]$ we have $Q_{ab}=\langle s_a s_b\rangle$ and therefore $P_{ab}=\delta_{ab}+Q_{ab}$. The above expression is valid at the third order in Q_{ab}^{ij}, which is enough to get the first nonlinear term in $\Phi(\alpha)$. The diagonal structure above has been obtained performing various manipulations on the original integral. In particular the matrix P enters only though its eigenvalues because the relevant expressions are rotationally invariant at the order considered and the integral in $S[Q,\alpha]$ is invariant under a simultaneous rotation of all the blocks Q^{ij}. The parameter p has been introduced for later convenience and has to be put to zero eventually (it has the physical meaning of adding a small perturbation on the couplings of replicas in different blocks effectively removing the degeneracy of permutations among them). Note that in the previous expression the off-diagonal indices run from 1 to $-N/n$ because we have exploited the fact that the function $S[Q,\alpha]$ in the thermodynamic limit is invariant under a rescaling $\{\alpha\to b\alpha, N\to bN\}$.

The above expression is suitable for an expansion in powers of α. The first $O(\alpha^2)$ term is obtained performing the Gaussian integral,

$$S[Q,\alpha] = \frac{\alpha^2}{4n^2}\sum_{ab}\ln(1-\beta^2\lambda_a\lambda_b+p)+O(\alpha^3). \quad (5)$$

In the limit $\alpha\to 0$ the extremum of $\Phi[Q,\alpha]$ is given by the extremum of $F[Q]$ with the first $O(\alpha^2)$ correction Eq. (5) evaluated on the free solution where $P_{ab}=\delta_{ab}+Q_{ab}$. In the $n\to 0$ limit the Gaussian correction turns out to be divergent if $p=0$, this is because the lowest eigenvalue λ_0 of P_{ab} obeys the following relationship: $\lambda_0\equiv 1-\int_0^1 q(x)dx=1/\beta$ that holds exactly at all temperatures.[12] The divergence of the $O(\alpha^2)$ correction suggests that the first nonlinear term in $\Phi(\alpha)$ has a power smaller than two. We also expect that the series in powers of α will be divergent at all orders and we will have to resum it in some way.

Before going into the next step of the computation, we consider a similar treatment of the spherical model. At zero temperature the energy is minus the largest eigenvalue of a random Gaussian $N\times N$ matrix. The logarithm of the probability that the largest eigenvalue is lower than its typical value is indeed $O(N^2)$ and has been recently computed.[3] These results tell us that in the zero-temperature limit we have: $\Phi(\alpha)=-\alpha\beta-\frac{2}{3}|\alpha\beta|^{3/2}+o(\alpha^{3/2})$. Repeating the above procedure we obtain that for the spherical model $\Phi(\alpha)$ can be computed as the saddle-point value of a functional similar to Eq. (2). The first $O(\alpha)$ term depends on a symmetric $n\times n$ matrix Q_{ab} (not vanishing on the diagonal) and on a parameter z enforcing the spherical constraint,

$$F[Q,z] = \frac{\beta}{4n}\sum_{ab}Q_{ab}^2-\frac{z}{\beta}+\frac{1}{2\beta n}\mathrm{Tr}\ln\left[zI-\frac{\beta^2}{2}Q\right], \quad (6)$$

where I is the identity matrix. The second term is equal (at the third order in Q_{ab}^{ij}) to expression (4) provided we have (in matrix notation) $P=(2zI-\beta^2Q)^{-1}$. The solution of the spherical model in the $\alpha\to 0$ limit is given by a RS $n\times n$ matrix $Q_{ab}\equiv q=1-T$ (Ref. 1) and again the lowest eigenvalue λ_0 of P turns out to obey the relationship $\lambda_0=1/\beta$ leading to a divergent $O(\alpha^2)$ correction.

We now face the task of computing the leading corrections that diverges in the $p\to 0$ limit. This can be done by studying the diagrammatic loop expansion of $S[Q,\alpha]$. This is a very technical process and we move it to the Appendix. A careful analysis of the diagrams at all orders yields the following two expressions for the leading behavior of $S[Q,\alpha]$ at small α,

$$S(Q,\alpha) = [-\alpha\beta\sqrt{T\dot{q}(0)}]^{12/7}C_{fRSB}+o(-\alpha)^{12/7},$$

$$S(Q,\alpha) = (-\alpha\beta q)^{3/2}C_{RS}+o(-\alpha)^{3/2}, \quad (7)$$

where

$$C_{fRSB}\equiv\lim_{z\to\infty}z^{2/7}f_{fRSB}(z), \quad (8)$$

$$C_{RS}\equiv\lim_{z\to\infty}z^{1/2}f_{RS}(z), \quad (9)$$

The loop expansion yields the series for the two functions $f_{fRSB}(z)$ and $f_{RS}(z)$ in powers of z. We have computed the series at fourth loop order,

$$f_{RSB}(z) = -\frac{\pi}{8}+0.456z-3.278z^2+36.11z^3+O(x^4),$$

$$f_{RS}(z) = -\frac{1}{4}+\frac{7}{6}z-19z^2+443z^3+O(z^4).$$

The problem of extracting the $z\to\infty$ behavior of functions like $f_{RSB}(z)$ and $f_{RS}(z)$ from their expansions in powers of z has already appeared in spin-glass literature[13] in a similar context and various resummation methods have been devised. Performing similar analysis on the above series we obtained the following estimates, $C_{RS}=-.68$, $C_{fRSB}=-.66$ with a 15% error. As a consequence the estimate for C_{RS} is in good agreement with the exact result $C_{RS}=-2/3$ quoted above.

In the fRSB case we obtain from (A13)

$$\frac{1}{N^2}\ln P(\Delta f) = -a_+[T\dot{q}(0)]^{-6/5}\Delta f^{12/5}+o(\Delta f^{12/5}). \quad (10)$$

Where $a_+\equiv(-C_{fRSB})^{-7/5}35(7/3)^{2/5}/(2^{4/5}144)$. The numerical data of Ref. 10 at zero temperature are highly consistent with the 12/5 scaling that we have obtained, see Fig. 1; a linear fit on the data corresponding to $N=150$ combined with

$\lim_{T\to 0} T\dot{q}(0) = .743$ (Ref. 2) leads to a numerical estimate $C_{fRSB} = -0.64(2)$ consistent with our prediction from the resummation of the series.

Remarkably the previous results display a great deal of universality. Indeed the exponents 12/7 and 3/2 and the coefficients C_{fRSB} and C_{RS} depends only on the fRSB or RS structure of the eigenvalues of the matrix P. The sole dependence on the actual model and on the temperature being, respectively, through the parameters $\sqrt{T\dot{q}(0)}$ and q. Interestingly enough the same universal behavior is displayed by the function $L(\Delta f)$, again both for RS and fRSB.[2] For both $L_2(\Delta f)$ and $L(\Delta f)$ we expect the universal behavior to hold only at the leading order for small Δf, nevertheless this points toward universality of the corresponding small-deviations distribution. Indeed the reader may have already noticed that the 12/5 exponent in Eq. (10) is fully consistent with the assumptions that the large positive deviations match the far right tail of the distribution of the small-deviations of the free energy density if they scales as $f - f_N = O(N^{1/6})$,[11] where f_N is the sample average at size N. At present the latter hypothesis is widely accepted in the literature after many recent numerical and theoretical investigations (see, e.g., Refs. 2, 7, and 10) and our findings add further support to it.

We conjecture that the probability distribution $P(\delta)$ of the of the rescaled variable,

$$\delta = (f - f_N)N^{5/6}/\sqrt{T\dot{q}(0)}, \tag{11}$$

is of the form $P(\delta) = \exp[-G(\delta)]$ where the large δ behavior of $G(\delta)$ matches with the behavior near the origins of the functions $L_2(\Delta f)$ and $L(\Delta f)$, i.e.,

$$G(\delta) \simeq a_- |\delta|^{6/5} \quad \text{for} \quad \delta \to -\infty, \tag{12}$$

$$G(\delta) \simeq a_+ \delta^{12/5} \quad \text{for} \quad \delta \to +\infty, \tag{13}$$

with $a_- = 1.366$ (Ref. 2) and $a_+ = 0.36$.

It is natural to assume that for a large class fRSB spin-glass model not only the large δ behavior of $G(\delta)$ is independent from of the model, but that the function $G(\delta)$ does not depend on the model for all δ and it is the same at all temperatures below the critical one. A similar results should be valid for replica symmetric models where the equivalent function $H[(f - f_N)N^{2/3}/q]$ is given by Tracy–Widom function.[4]

It is important to remember that the present conjectures are meant to apply only for models whose large deviations are described by an action of the form (4). On the other hand, it has been shown that, in general, spin-glass models defined on random lattices present trivial Gaussian fluctuations of the free energy.[15] It may also be noted that the present treatment does not apply to the fully connected SK model with m-component spins in the large m limit. Although this model is RS, the analysis of its large devation behavior requies one to treat carefully the large m limit, making even the computation of large deviations for positive values of number of replicas nontrivial (the fluctuations of the ground state energy were claimed to scale as $N^{1/5}$ by Hastings[16] and this result has been recently rederived through a different technique[17]).

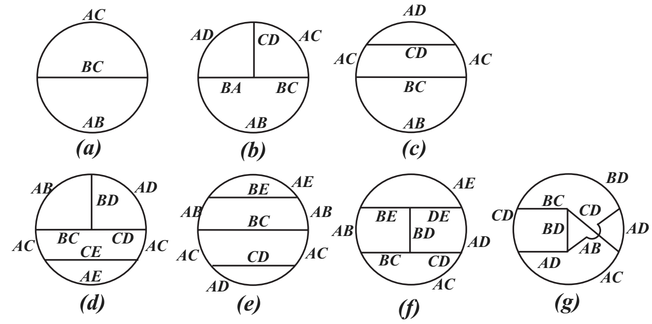

FIG. 2. Indexed Graphs of the cubic theory up to four loops. Note that each index must belong to a closed path on the graph. The first six graphs are relevant because they satisfy the condition $K = L+1$ while graph (g) is irrelevant because $K = 4 < L+1 = 5$ and therefore it yields an $O(N^{-1})$ contribution.

APPENDIX: THE LOOP EXPANSION

In this appendix we present the analysis of the loop expansion of $S[Q,\alpha]$. We introduce capital indices that corresponds to a couple of upper and lower indices $A = \binom{i}{a}$. Given a graph with vertices of degree three we have to associate to each line in it a couple of *different* capital indices in such a way that each vertex of the graph corresponds to a term in the action of Eq. (4). This limits the number of K free indices and for a graphs with L loops one can prove that $K \leq L+1$. Valid indexed graphs G_I at four loops are represented in Fig. 2. For a graph G_I with V vertices we have a factor $-(-\alpha/N)^{V/2+2}/V!$ times a factor $\beta^2\lambda_a\lambda_b/(p+1-\beta^2\lambda_a\lambda_b)$ where ab are the lower indices on each line. We sum over all the free K upper indices (they can take values $1,\ldots,-N/n$) and vertex permutations of the graph. This yields a factor $V!(-N/n)^K$ that has to be divided by an appropriate symmetry factor $M(G_I)$ to avoid overcounting. Relevant graphs are those that yield an $O(N^2)$ contribution, i.e., they satisfy the condition $K = V/2+2$ or equivalently $K = L+1$. Consistently it can be shown that no graph can yield a contribution greater than $O(N^2)$. Summarizing the sum over the upper indices yields the factor,

$$\frac{(-\alpha)^K(-1)^{K+1}}{M(G_I)n^K}, \tag{A1}$$

that multiplies the result coming from the sum over the lower indices that we analyze now. In order to sum over the lower indices we have to take care of the non trivial structure of the propagator. In particular given an indexed graph each line with lower indices ab in the graph corresponds to a factor $\beta^2\lambda_a\lambda_b/(p+1-\beta^2\lambda_a\lambda_b)$. The resulting object has to be summed over the K free-lower indices (each running from 1 to n) leading to,

$$\frac{1}{n^K}\sum_{a_1,\ldots,a_K}^{n}\prod_{lines}\frac{\beta^2\lambda_{a_i}\lambda_{a_j}}{p+1-\beta^2\lambda_{a_i}\lambda_{a_j}} \tag{A2}$$

where we have borrowed the factor n^{-K} from Eq. (A1).

To clarify the procedure we consider graph (a) in Fig. 2. According to the above expression the factor that we need to compute is

$$\lim_{n\to 0}\frac{1}{n^3}\sum_{abc}\frac{\beta^2\lambda_a\lambda_b}{p+1-\beta^2\lambda_a\lambda_b}\frac{\beta^2\lambda_b\lambda_c}{p+1-\beta^2\lambda_b\lambda_c}\frac{\beta^2\lambda_a\lambda_c}{p+1-\beta^2\lambda_a\lambda_c}. \tag{A3}$$

To perform this summation we must use some well know results on the eigenvalues of a replica symmetric matrix Q and a hierarchical matrix Q_{ab} characterized by a function $q(x)$.[14] In the RS case we have two eigenvalues

$$\lambda_0 = q_d + (n-1)q \quad \text{deg:}1,$$

$$\lambda_1 = q_d - q = \lambda_0 - nq \quad \text{deg:}n-1.$$

In the full RSB case we have instead,

$$\lambda_0 = q_d - \int_n^1 q(x)dx \quad \text{deg:}1,$$

$$\lambda_x = q_d - [xq(x) + \int_x^1 q(x)dx] \quad \text{deg:}-n\frac{dx}{x^2},$$

where the relationship $-x\dot{q}(x)=\dot{\lambda}_x$ holds. According to the above expressions the sum over the eigenvalues of a given function $f(\lambda_a)$ is given in the $n\to 0$ limit by

$$\lim_{n\to 0}\frac{1}{n}\sum_a f(\lambda_a) = f(\lambda_0) + q\left.\frac{df}{d\lambda}\right|_{\lambda=\lambda_0}, \tag{A4}$$

$$\lim_{n\to 0}\frac{1}{n}\sum_a f(\lambda_a) = f(\lambda_1) + \int_0^1 \dot{q}(y)\left.\frac{df}{d\lambda}\right|_{\lambda=\lambda_y} dy, \tag{A5}$$

Respectively, for the RS and fRSB case. Now we consider the RS case. We rewrite the line factor Eq. (A2) changing variable from $\beta\lambda_a=\beta\lambda_0+\beta\Delta\lambda_a=1+\beta\Delta\lambda_a$ and make the substitution $\beta\Delta\lambda_a=px_a$, at leading order in p we have

$$\frac{\beta^2\lambda_a\lambda_b}{p+1-\beta^2\lambda_a\lambda_b} \to \frac{1}{p}\times\frac{1}{1-x_a-x_b}. \tag{A6}$$

All these factors have to be summed over the K free indices, with the most diverging contribution in p coming from the second term in the r.h.s. of Eq. (A4). For graph (a) in Fig. 2 we have that the most diverging term in Eq. (A3) is given by

$$\frac{(\beta q)^3}{p^6}\frac{d^3}{dx_a dx_b dx_c}\left.\left(\frac{1}{1-x_a-x_b}\times\frac{1}{1-x_b-x_c}\times\frac{1}{1-x_a-x_c}\right)\right|_{x_a=x_b=x_c=0} = 14\frac{(\beta q)^3}{p^6}. \tag{A7}$$

The above expression can be easily generalized to compute the factor Eq. (A2) for a general graph. The final result at leading order in p for the RS case is

$$\frac{(\beta q)^K}{p^{K+I}}\frac{d^K}{dx_1\cdots dx_K}\left.\prod_{lines}\frac{1}{1-x_a-x_b}\right|_{x_1=\ldots=x_K=0}. \tag{A8}$$

In the full RSB case we first observe that the behavior of $\beta\lambda_y$ at small values of y can be obtained from the relationship $-y\dot{q}(y)=\dot{\lambda}_y$,

$$\beta\lambda_y = 1 - \frac{\dot{q}(0)\beta}{2}y^2 + O(y^4). \tag{A9}$$

Then we rewrite the line factor changing variable from $\beta\lambda_y=\beta\lambda_0+\beta\Delta\lambda_y\lambda_a=1+\beta\Delta\lambda_y$ and we rescale the variable y in such a way that $\dot{q}(0)\beta y^2=x^2$, then we have at leading order in p,

$$\frac{\beta^2\lambda_a\lambda_b}{p+1-\beta^2\lambda_a\lambda_b} \to \frac{1}{p}\times\frac{1}{1+\frac{1}{2}x_a^2+\frac{1}{2}x_b^2}. \tag{A10}$$

Much as in the RS case the sum over the K indices is dominated by the second term in the r.h.s. of Eq. (A5). We introduce the following operator defined on a function of $f(x)$:

$$A_x[f[x]] \equiv -\int_0^\infty \frac{1}{x}\frac{df}{dx}dx. \tag{A11}$$

Now the factor Eq. (A3) of graph (a) at leading divergent order reads,

$$\frac{[\beta\sqrt{T\dot{q}(0)}]^3}{p^{9/2}}A_{x_a}\circ A_{x_b}\circ A_{x_c}\circ,$$

$$\left[\frac{1}{\left(1+\frac{1}{2}x_a^2+\frac{1}{2}x_b^2\right)\left(1+\frac{1}{2}x_b^2+\frac{1}{2}x_c^2\right)\left(1+\frac{1}{2}x_a^2+\frac{1}{2}x_c^2\right)}\right]$$

$$= \frac{[\beta\sqrt{T\dot{q}(0)}]^3}{p^{9/2}}5.47788.$$

The above expression can be easily extended for a generic graph, we must multiply for each line ab in the graph a factor $1/(1+x_a^2/2+x_b^2/2)$ and apply the operator A_a for each of the K indices. The most diverging contribution in p is given in the full-RSB case by,

$$\frac{[\beta\sqrt{T\dot{q}(0)}]^K}{p^{K/2+I}} A_{x_1} \circ \cdots \circ A_{x_K} \circ \prod_{lines} \frac{1}{1+\frac{1}{2}x_a^2+\frac{1}{2}x_b^2}. \tag{A12}$$

As we already noticed a relevant diagram has $K=L+1$ therefore in the cubic theory we have $I=3K-6$. Rescaling the variable p, respectively, as $p=[-\alpha\beta\sqrt{T\dot{q}(0)}/z]^{2/7}$, $p=(-\alpha\beta q/z)^{1/4}$ and multiplying by the factors coming from Eq. (A1) we get an expression of $S[Q,\alpha]$ in terms of two functions $f_{RS}(z)$ and $f_{fRSB}(z)$. The loop expansion yields the series of the two functions in powers of z. Taking the $z\to\infty$ limit we eventually obtain,

$$S(Q,\alpha) = [-\alpha\beta\sqrt{T\dot{q}(0)}]^{12/7} C_{fRSB} + o(-\alpha)^{12/7},$$

$$S[Q,\alpha] = (-\alpha\beta q)^{3/2} C_{RS} + o(-\alpha)^{3/2}, \tag{A13}$$

where $C_{fRSB} \equiv \lim_{z\to\infty} z^{2/7} f_{fRSB}(z)$ and $C_{RS} \equiv \lim_{z\to\infty} z^{1/2} f_{RS}(z)$. At fourth loop order the diagrams are shown in Fig. 2 and lead to the following expressions:

$$f_{RSB}(z) = -\frac{\pi}{8} + 0.456z - 3.278z^2 + 36.11z^3 + O(x^4),$$

$$f_{RS}(z) = -\frac{1}{4} + \frac{7}{6}z - 19z^2 + 443z^3 + O(z^4).$$

The zeroth order terms comes from a similar treatment of the Gaussian contribution Eq. (5).

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